

Problem 2.33

[Difficulty: 3]

2.33 A flow is described by velocity field $\vec{V} = ax\hat{i} + b\hat{j}$, where $a = 1/5 \text{ s}^{-1}$ and $b = 1 \text{ m/s}$. Coordinates are measured in meters. Obtain the equation for the streamline passing through point (1, 1). At $t = 5 \text{ s}$, what are the coordinates of the particle that initially (at $t = 0$) passed through point (1, 1)? What are its coordinates at $t = 10 \text{ s}$? Plot the streamline and the initial, 5 s, and 10 s positions of the particle. What conclusions can you draw about the pathline, streamline, and streakline for this flow?

Given: Velocity field

Find: Equation for streamline through point (1,1); coordinates of particle at $t = 5 \text{ s}$ and $t = 10 \text{ s}$ that was at (1,1) at $t = 0$; compare pathline, streamline, streakline

Solution:

Governing equations: For streamlines $\frac{v}{u} = \frac{dy}{dx}$ For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$

Assumption: 2D flow

Given data $a = \frac{1}{5} \frac{1}{\text{s}}$ $b = 1 \frac{\text{m}}{\text{s}}$ $x_0 = 1$ $y_0 = 1$ $t_0 = 0$

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{b}{a \cdot x}$

So, separating variables $\frac{a}{b} \cdot dy = \frac{dx}{x}$

Integrating $\frac{a}{b} \cdot (y - y_0) = \ln\left(\frac{x}{x_0}\right)$

The solution is then $y = y_0 + \frac{b}{a} \cdot \ln\left(\frac{x}{x_0}\right) = 5 \cdot \ln(x) + 1$

Hence for pathlines $u_p = \frac{dx}{dt} = a \cdot x$ $v_p = \frac{dy}{dt} = b$

Hence $\frac{dx}{x} = a \cdot dt$ $dy = b \cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = a \cdot (t - t_0)$ $y - y_0 = b \cdot (t - t_0)$

The pathlines are $x = x_0 \cdot e^{a \cdot (t - t_0)}$ $y = y_0 + b \cdot (t - t_0)$ or $y = y_0 + \frac{b}{a} \cdot \ln\left(\frac{x}{x_0}\right)$

For a particle that was at $x_0 = 1$ m, $y_0 = 1$ m at $t_0 = 0$ s, at time $t = 1$ s we find the position is

$$x = x_0 \cdot e^{a \cdot (t - t_0)} = e^{\frac{1}{5}} \text{ m} \quad y = y_0 + b \cdot (t - t_0) = 2 \text{ m}$$

For a particle that was at $x_0 = 1$ m, $y_0 = 1$ m at $t_0 = 0$ s, at time $t = 5$ s we find the position is

$$x = x_0 \cdot e^{a \cdot (t - t_0)} = e \text{ m} \quad y = y_0 + b \cdot (t - t_0) = 6 \text{ m}$$

For a particle that was at $x_0 = 1$ m, $y_0 = 1$ at $t_0 = 0$ s, at time $t = 10$ s we find the position is

$$x = x_0 \cdot e^{a \cdot (t - t_0)} = e^2 \text{ m} \quad y = y_0 + b \cdot (t - t_0) = 11 \text{ m}$$

For this steady flow streamlines, streaklines and pathlines coincide

